Докл. Акад. Наук СССР
Том 282 (1985), № 3

Soviet Math. Dokl.
Vol. 31 (1985), No. 3

ON POISSON BRACKETS AND ONE-DIMENSIONAL HAMILTONIAN SYSTEMS OF HYDRODYNAMIC TYPE

UDC 513.835

S. P. TSAREV

The following definitions were introduced in [1] in connection with the investigation of so-called equations of slow modulations of Whitham type.

DEFINITION 1. A Poisson bracket on the space of field variables $u^k(x)$ is called a *bracket of hydrodynamic type* if it has the form

$$\{u^i(x), u^j(x)\} = g^{ij}(u(x))\delta'(x-y) + b_k^{ij}(u(x))u_x^k\delta(x-y).$$

DEFINITION 2. *Hamiltonians of hydrodynamic type* are defined to be functionals of the form $H = \int h(u)\,dx$, where the density $h(u)$ does not depend on the derivatives $u_x, u_{xx}, \ldots$.

In this case the Hamiltonian flows generated are given by equations of hydrodynamic type

$$(1)\qquad u_t^i = v_j^i(u)u_x^j, \qquad i,j = 1,\ldots,n.$$

According to Theorem 1 of [1], in the case of a nondegenerate matrix g^{ij}, which is the only case considered below, the conditions $g^{ij} = g^{ji}$ and $b_k^{ij} = g^{is}\Gamma_{sk}^j$ must be satisfied, where Γ_{sk}^j is the symmetric connection generated by g^{ij} whose curvature tensor is equal to zero. From this we see that $v_j^i = \nabla^i\nabla_j h$; the covariant derivatives are taken by means of Γ_{jk}^i. We shall investigate the structure of the matrices v_j^i, which are henceforth called *Hamiltonian matrices*. For simplicity we assume that all eigenvalues of v_j^i are real and distinct.

The following assertions are obvious.

LEMMA 1. *In order that the matrix $v_j^i(u)$ be Hamiltonian it is necesary and sufficient that there exists a nondegenerate metric $g^{ij}(u)$ of zero curvature such that*

a) $g^{ik}v_k^j = g^{jk}v_k^i$, *and*

b) $\nabla^i v_k^j = \nabla^j v_k^i$, *where the ∇^i are the covariant derivatives given by the matric g^{ij}.*

Under the change $x \leftrightarrow t$ equation (1) is transformed into $u_t^i = (v^{-1})_j^i u_x^j$, where v^{-1} is the matrix inverse to v_j^i.

THEOREM 1. *Let v_j^i be a nondegenerate Hamiltonian matrix. Then the inverse matrix $(v^{-1})_j^i$ is also Hamiltonian, and the metric corresponding to it has the form $g'_{ij} = v_i^k g_{kl} v_j^l$, while the Hamiltonian h' considered in coordinates flat relative to g'_{ij} is the Legendre transform of the Hamiltonian h considered in coordinates flat relative to g_{ij}.*

COROLLARY. *Under any linear changes of the variables (x,t) a Hamiltonian system is transformed into a Hamiltonian system.*

The corollary follows from Theorem 1 and the fact that under such a change v_j^i goes over into $a\delta_j^i + b((v+cE)^{-1})_j^i$.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35Q20, 76B99.

Below we shall study Hamiltonian systems with a number of field variables $n \geq 3$. The special case $n = 2$ has been studied in detail by M. Polyak.

REMARK. If v_j^i is a Hamiltonian matrix of "general position", then the metric corresponding to it can be recovered uniquely up to multiplication by a global constant.

PROBLEM (S. P. Novikov). Investigate Hamiltonian systems with a diagonal matrix $v_j^i(u)$. According to a conjecture of Novikov, these systems possess higher integrability, in particular, "superfluous" conservation laws. The study of Hamiltonian systems diagonal in some coordinate system u^i is of special interest. For example, the average of the Korteweg-deVries equation is such a system (see [3]); for finite-zone solutions (see [1]) diagonality of averaged equations of Whitham type is established in [4].

Let $v_k^i = v_k\delta_k^i$, where the v_k are diagonal coefficients. For simplicity we assume that all the v_k are distinct. By Lemma 1 this implies that the metric g^{ij} is diagonal in the same coordinates, and also

$$(2)\qquad \partial_k v_k/(v_k - v_i) = \Gamma_{ik}^i, \qquad \partial_k = \partial/\partial u_k$$

($i \neq k$; there is no summation over repeated indices). From this, since for a diagonal metric $\Gamma_{ik}^i = \partial_k \ln\sqrt{g_{ii}}$, it follows that

$$\partial_j(\partial_k v_i/(v_k - v_i)) = \partial_k(\partial_j v_i/(v_j - v_i))$$

(i, j and k are distinct indices; there is no summation).

From the point of view of differential geometry prescription of a diagonal metric of zero curvature is the prescription of a curvilinear orthogonal coordinate system in flat space (the metric in our case may have variable sign), i.e., a coordinate system with mutually orthogonal coordinate surfaces or n one-parameter families of mutually orthogonal $(n-1)$-dimensional hypersurfaces. Such coordinate systems were studied in detail in classical differential geometry back in the nineteenth century. In particular, such coordinate systems are defined locally by prescribing $n(n-1)/2$ functions of two variables (see [2]).

THEOREM 2. *Suppose there is given a curvilinear orthogonal coordinate system in $\mathbf{R}^n$, i.e., a diagonal metric of zero curvature. Then the system of equations (2) at the function $v_i(u)$ satisfies the consistency conditions and hence has a solution given by the values of v_i on the ith coordinate axis of the orthogonal coordinate system.*

Thus, a diagonal Hamiltonian system is determined in which $n(n-1)/2$ functions of two variables are arbitrary and n functions of a single variable are arbitrary.

We shall consider the question of the existence for Hamiltonian systems (1) of the integrals of hydrodynamic type $I = \int P(u)\,dx$, i.e., integrals not depending on the derivatives $u_x, u_{xx}, \ldots$. In this case the necessary and sufficient condition that I is an integral of (1) is $(\nabla_k\nabla_l P)v_j^l = (\nabla_j\nabla_l P)v_k^l$, where the covariant derivatives are taken by means of the metric in which (1) is Hamiltonian. This condition is equivalent to the commutativity of the two Hamiltonian matrices $\nabla^i\nabla_j P$ and $v_j^i = \nabla^i\nabla_j h$; hence, if the diagonal elements are distinct, both matrices can be diagonalized simultaneously. If v_j^i is a diagonal matrix in some region, then the diagonal elements $p_i(u)$ of the matrix $\nabla^i\nabla_j P$ (which must also be diagonal in this coordinate system!) satisfy the same relations as the v_i, namely (2).

From Theorem 2 we obviously have

THEOREM 3. *A diagonal Hamiltonian system always has infinitely many integrals of hydrodynamic type $I = \int P(u)\,dx$ in involution, n functions of a single variable arbitrary. They generate commuting diagonal Hamiltonian systems associated with one fixed curvilinear n-orthogonal coordinate system.*

Suppose the system (1) is not diagonalizable. At each point of the space of field variables u^i we choose a g-orthonormal basis $\{e_i\}$ of eigenvectors of v_j^i

$$(e_i, e_j) = \varepsilon_i\delta_{ij}, \qquad \varepsilon_i = \pm 1,$$

and we let $[e_i, e_j] = c^k_{ij} e_k$. We assume that our system is of general position, i.e., all coefficients c^k_{ij} with distinct i, j, k are nonzero. It can be shown that for $n = 3$ this is always the case for a nondiagonalizable system. In this case we obtain from Lemma 1 a collection of relations for the eigenvalues of the matrices v^i_j and $\nabla^i \nabla_j P$, from which it follows that $p_i = cv_i + d$ where c and d are constants. If we now go over to a flat coordinate system in which

$$g_{ij} = \varepsilon_i \delta_{ij}, \qquad v^i_j = \varepsilon_i \partial_i \partial_j h, \qquad \nabla^i \nabla_j P = \varepsilon_i \partial_i \partial_j P,$$

then we find that

$$P = ch + d \sum \varepsilon_i (u^i)^2 + \sum a_i u^i + b.$$

We have thus proved the following result.

THEOREM 4. *A general nondiagonalizable system has only linear combinations of the following integrals as hydrodynamic integrals* $I = \int P(u)\,dx$:

a) *the Hamiltonian of the system*;

b) *the square of the line element in flat coordinates*; *and*

c) *a linear function of the flat coordinates plus a constant.*

We shall show that the integrals obtained suffice for the integration of a diagonal Hamiltonian system. We choose a level surface of all integrals found in Theorem 3 which contains an "admissible" function $u^i(x)$, i.e., such that

a) each component of $u^i(x)$ assumes any value at most twice; and

b) $u^i_{xx} \neq 0$ at points where $u^i_x = 0$ (the set of such functions is open in the space of smooth functions on the line).

THEOREM 5 (COMPLETENESS THEOREM). *For any variation* δu^i *of an admissible function tangent to a level surface of all integrals of hydrodynamic type constructed in Theorem* 3 *there is one of them—the integral* $I_1 = \int P_1(u)\,dx$*—such that* $\delta u^i = (\nabla^i \nabla_j P_1) u^j_x$. *This means that a "tangent vector" to a level surface of this family of integrals at an admissible point* $u^i(x)$ *is a vector of the Hamiltonian flow generated by the integral* I_1.

A generalized hodograph method (an analogue of the method of the inverse problem for averaged equations). We shal present an explicit procedure for integrating an arbitrary diagonal Hamiltonian system. Let $v_i(u)$ be the diagonal coefficients of the matrix of the system. We consider the linear system (2) for finding the coefficients of flows commuting with (1) which correspond to hydrodynamic integrals of the given system, and we let $w_i(u)$ be some solution of it (with n functions of a single variable arbitrary). For given x and t we write the system of n equations with n unknowns u^i:

$$w_i(u) = v_i(u)t + x. \tag{3}$$

THEOREM 6. *A solution* $u^i(x,t)$ *of the system* (3) *satisfies the original system* (1). *Any solution of* (1) *in a neighborhood of a point* (x_0, t_0) *at which the derivatives* $u^i_x(x_0, t_0) \neq 0$ *can be obtained by this procedure.*

COROLLARY. *For the averaged Korteweg-de Vries equations* (*the Whitham equations*) *linear combinations of the first n averaged Kruskal integrals* (*see* [1]) *are involutive and generate the collection of exact solutions according to Theorem* 6.

We call these solutions *averaged n-zone solutions*, and we shall study them in more detail elsewhere.

Apparently, this purely local procedure is in a certain sense analogous to the local procedure of [5] for the equations of a principal chiral field and the sine-Gordon equation, where it was necessary to solve a Riemann problem in place of the simple system of equations (3)

Moscow State University

Received 16/MAY/84

BIBLIOGRAPHY

1. B. A. Dubrovin and S. P. Novikov, Dokl. Akad. Nauk SSSR **270** (1983), 781–785; English transl. in Soviet Math. Dokl. **27** (1983).

2. Élie Cartan, *Les systèmes différentials extérieurs et leurs applications géométriques*, Actualités Sci. Indust., no. 994, Hermann, Paris, 1945.

3. G. B. Whitham, *Linear and nonlinear waves*, Wiley, 1974.

4. H. Flaschka, M. G. Forest and D. W. McLaughlin, Comm. Pure. Appl. Math. **33** (1980), 739–784.

5. I. M. Krichever, Dokl. Akad. Nauk SSSR **253** (1980), 288–292; English transl., in Soviet Math. Dokl. **21** (1980).

Translated by J. R. SCHULENBERGER