Modification of a Two-Dimensional Fast Fourier Transform Algorithm by the Analog of the Cooley–Tukey Algorithm for a Rectangular Signal[1]

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Abstract—One-dimensional fast Fourier transform (FFT) is the most popular tool for computing the two-dimensional Fourier transform. As a rule, a standard method of combination of one-dimensional FFTs—the so-called algorithm "by rows and columns" [1]—is used in the literature. In [2, 3], the authors showed how to compute the FFT for a signal with the number of samples $2^s \times 2^s$ with the use of an analog of the Cooley–Tukey algorithm. In the present paper, a two-dimensional analog of the Cooley–Tukey algorithm is constructed for a rectangular signal with the number of samples $2^s \times 2^{s+\vartheta}$. The number of operations in this algorithm is much less than that in the successive application of a one dimensional FFT by rows and columns. The testing of the algorithm on image-type signals shows that the speed of computation of the FFT by the algorithm proposed is about 1.7 times higher than that of the algorithm by rows and columns.

Keywords: two-dimensional fast Fourier transform, Cooley-Tukey FFT, parallel FFT algorithm.

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1. DESCRIPTION OF THE ALGORITHM

Consider a signal f that represents a two-dimensional periodic signal with period 2^s in the first and $2^{s+\vartheta}$ in the second coordinates. Samples are defined as $f_{k,t}$, where $k = 0 : 2^s - 1$ and $t = 0 : 2^{s+\vartheta} - 1$. A discrete Fourier transform (DFT) for the signal f is defined by the formula

$$F_{l,m} = \sum_{k=0}^{2^s-1}\sum_{t=0}^{2^{s+\vartheta}-1} f_{k,t} e^{\frac{2\pi ilk}{2^s}} e^{\frac{2\pi imt}{2^{s+\vartheta}}}. \tag{1}$$

A two-dimensional DFT can be computed by a combination of one-dimensional DFTs. To this end, one computes F in the following form:

$$F_{l,m} = \sum_{k=0}^{2^s-1}\left[\sum_{t=0}^{2^{s+\vartheta}-1} f_{k,t} e^{\frac{2\pi ilk}{2^s}}\right] e^{\frac{2\pi imt}{2^{s+\vartheta}}}. \tag{2}$$

The sum in square brackets in (2) represents the computation of a one-dimensional DFT, for example, by rows; then the outer sum represents the computation of a one-dimensional DFT by columns. Let us transform this formula by dividing the second coordinate into even and odd components:

$$\begin{aligned} F_{l,m} &= \sum_{t=0}^{2^{s+\vartheta}-1}\left[\sum_{k=0}^{2^s-1} f_{k,t} e^{\frac{2\pi ilk}{2^s}}\right] e^{\frac{2\pi imt}{2^{s+\vartheta}}} \\ &= \sum_{t_1=0}^{2^{s+\vartheta-1}-1}\left[\sum_{k=0}^{2^s-1} f_{k,2t_1} e^{\frac{2\pi ilk}{2^s}}\right] e^{\frac{2\pi im2t_1}{2^{s+\vartheta}}} \\ &+ \sum_{t_1=0}^{2^{s+\vartheta-1}-1}\left[\sum_{k=0}^{2^s-1} f_{k,2t_1+1} e^{\frac{2\pi ilk}{2^s}}\right] e^{\frac{2\pi im(2t_1+1)}{2^{s+\vartheta}}} \\ &= \sum_{t_1=0}^{2^{s+\vartheta-1}-1}\left[\sum_{k=0}^{2^s-1} f_{k,2t_1} e^{\frac{2\pi ilk}{2^s}}\right] e^{\frac{2\pi imt_1}{2^{s+\vartheta-1}}} \\ &+ \sum_{t_1=0}^{2^{s+\vartheta-1}-1}\left[\sum_{k=0}^{2^s-1} f_{k,2t_1+1} e^{\frac{2\pi ilk}{2^s}}\right] e^{\frac{2\pi imt_1}{2^{s+\vartheta-1}}} e^{\frac{2\pi im}{2^{s+\vartheta}}} \\ &= g^1_{0,0}(l,m) + e^{\frac{2\pi im}{2^{s+\vartheta}}} g^1_{0,1}(l,m), \end{aligned} \tag{3}$$

where $f_{k,2t_1}$ and $f_{k,2t_1+1}$ are two-dimensional subsignals of the signal $f_{k,t}$ that contain the components of the signal $f_{k,t}$ with even and odd indexes in the second coordinate, respectively. It is clear that the dimension of these signals is $2^s \times 2^{s+\vartheta-1}$. Notice that $g^1_{0,0}(l,m)$

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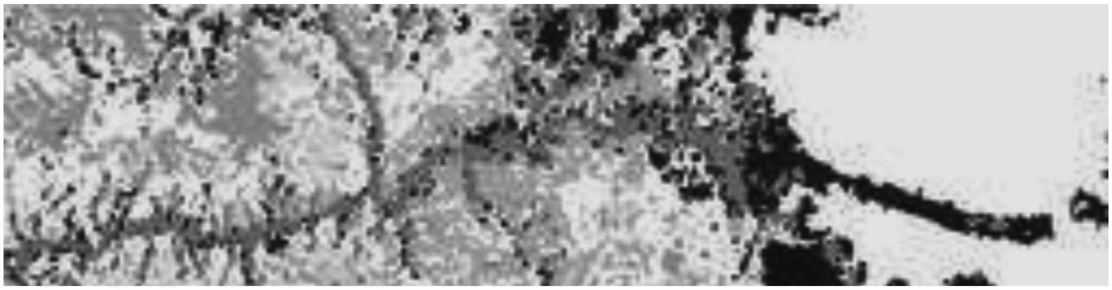

Example of a source signal with the number of samples 1024 × 4096.

and $g^1_{0,1}(l, m)$ are two-dimensional DFTs for the subsignals $f_{k,2t_1}$ and $f_{k,2t_1+1}$, respectively.

One can show that the multiplier $e^{\frac{2\pi i m}{2^{s+\vartheta}}}$ in formula (3) possesses the property of symmetry with respect to $2^{s+\vartheta-1}$; i.e., for $m_1 = 0 : 2^{s+\vartheta-1}$ and $m = m_1 + 2^{s+\vartheta-1}$, we have

$$e^{\frac{\pi i(2^{s+\vartheta-1}+m_1)}{2^{s+\vartheta}}} = e^{\frac{\pi i 2^{s+\vartheta-1}}{2^{s+\vartheta}}} e^{\frac{\pi i m_1}{2^{s+\vartheta}}} = e^{\frac{\pi i}{2}} e^{\frac{\pi i m_1}{2^{s+\vartheta}}} = -e^{\frac{\pi i m_1}{2^{s+\vartheta}}}. \quad (4)$$

For convenience, denote

$$F^0_{l,m} = F_{l,m}. \quad (5)$$

Then formulas (3) and (4) with regard to the notations (5) yield

$$\begin{aligned} F^0_{l,m_1} &= g^1_{0,0}(l, m_1) + e^{\frac{\pi i m_1}{2^{s+\vartheta}}} g^1_{0,1}(l, m_1), \\ F^0_{l,m_1+2^{s+\vartheta-1}} &= g^1_{0,0}(l, m_1 + 2^{s+\vartheta-1}) \\ &\quad - e^{\frac{\pi i m_1}{2^{s+\vartheta}}} g^1_{0,1}(l, m_1 + 2^{s+\vartheta-1}), \end{aligned} \quad (6)$$

where $l = 0 : 2^s - 1$ and $m_1 = 0 : 2^{s+\vartheta-1} - 1$.

For each of the sums $g^1_{0,0}(l, m_1)$ and $g^1_{0,1}(l, m_1)$, we can continue the procedure of division of the second coordinate into even and odd components similar to (3). We obtain four sums

$$g^2_{0,0}(l, m_1), \quad g^2_{0,1}(l, m_1), \quad g^2_{0,0}(l, m_1'), \quad g^2_{0,1}(l, m_1'), \quad (7)$$

where $l = 0 : 2^s - 1$, m_1, $m_1' = 0 : 2^{s+\vartheta-1} - 1$, m_1 are even components, and m_1' are odd components.

Next, applying procedure (3) to formulas (7), at step ϑ we obtain 2^ϑ sums of the form

$$\begin{aligned} g^\vartheta_{0,0}(l, m_{\vartheta-1}) &= \sum_{t_1=0}^{2^s-1}\left[\sum_{k=0}^{2^s-1} f^{\vartheta-1}_{k,2t_1} e^{\frac{2\pi i l k}{2^s}}\right] e^{\frac{2\pi i m_{\vartheta-1} t_1}{2^s}}, \\ g^\vartheta_{0,1}(l, m_{\vartheta-1}) &= \sum_{t_1=0}^{2^s-1}\left[\sum_{k=0}^{2^s-1} f^{\vartheta-1}_{k,2t_1+1} e^{\frac{2\pi i l k}{2^s}}\right] e^{\frac{2\pi i m_{\vartheta-1} t_1}{2^s}}, \end{aligned} \quad (8)$$

where $l = 0 : 2^s - 1$ and $m_{\vartheta-1} = 0 : 2^{s+1} - 1$. The sums (8) differ by the set of input samples $m_{\vartheta-1}$ obtained by the division of the set $m_{\vartheta-2}$ into even and odd components at the previous step.

Hence, after ϑ steps, we arrive at signals of dimension $2^s \times 2^s$ of the form

$$\begin{aligned} F^{\vartheta-1}_{l,m_\vartheta} &= g^\vartheta_{0,0}(l, m_\vartheta) + e^{\frac{\pi i m_\vartheta}{2^{s+1}}} g^\vartheta_{0,1}(l, m_\vartheta), \\ F^{\vartheta-1}_{l,m_\vartheta+2^s} &= g^\vartheta_{0,0}(l, m_\vartheta + 2^s) + e^{\frac{\pi i m_\vartheta}{2^{s+1}}} g^\vartheta_{0,1}(l, m_\vartheta + 2^s), \end{aligned} \quad (9)$$

where l, $m_\vartheta = 0 : 2^s - 1$.

Let us apply a two-dimensional FFT algorithm by the analog of the Cooley–Tukey algorithm described in [3] to each of 2^s signals (9). Population of the spectra of these signals is the discrete Fourier transform of the signal f.

Let us calculate the number of operations. Procedure (3) takes $2^{\vartheta-1}\vartheta$ operations of complex multiplications and $2^{\vartheta+1}\vartheta$ complex additions. The two-dimensional FFT by the analog of the Cooley–Tukey algorithm applied to finite signal (9) of dimension $2^s : 2^s$ takes $3 \cdot 2^{2s-s}s$ operations of complex multiplications and 2^{2s+1} operations of complex additions [3]. Then the total number of operations for processing the source signal $f_{k,t}$, where $k = 0 : 2^{s-1}$ and $t = 0 : 2^{s+\vartheta-1}$, is $3 \cdot 2^{2s+\vartheta-3}(s+\vartheta)$ operations of complex multiplications and $2^{2s+\vartheta+1}(s+\vartheta)$ operations of complex additions. The computation of the FFT of the source signal $f_{k,t}$ by division into rows and columns takes $2^{2s+\vartheta-1}(2s+\vartheta)$ operations of complex multiplication and $2^{2s+\vartheta-3}(2s+\vartheta)$ operations of complex addition.

2. THE RESULTS OBTAINED

To test the algorithm, we wrote a program in the C++ language with the use of the message passing interface (MPI) library. The testing was carried out on a personal computer with the following characteristics:

- Processor: Intel Core 2 Duo CPU T8100 2.1 GHz;
- Main memory: 2 Gb;
- Operating system: Windows XP Service Pack 3.

Table. Running time of a two-dimensional FFT algorithm for a rectangular signal (in seconds)

Height	Width	Number of processes	FFT by rows and columns		FFT by the analog of the Cooley–Tukey algorithm	
			2 processors	4 processors	2 processors	4 processors
1024	2048	1	1.105	1.081	0.666	0.645
		2	1.356	1.345	2.441	2.421
		4	0.986	1.123	2.121	1.903
		8	0.768	1.233	1.757	2.566
		16	11.322	1.101	11.900	71.360
	4096	1	2.414	2.392	1.413	1.410
		2	2.377	2.355	4.132	4.097
		4	1.686	2.556	3.111	7.130
		8	1.319	2.467	8.976	32.455
		16	–	4.414	–	51.223
	8192	1	6.057	5.970	3.312	3.848
		2	5.179	4.812	7.110	7.505
		4	3.671	4.677	6.500	13.001
		8	2.938	3.833	5.990	25.020
		16	–	7.600	–	–

The running time of the algorithms is presented in the table. An example of the source signal is shown in the figure.

3. CONCLUSIONS

A modified two-dimensional FFT algorithm by the analog of the Cooley–Tukey algorithm for a rectangular signal takes a smaller number of complex operations of addition and multiplication and runs faster than the analog of the two-dimensional FFT algorithm by rows and columns.

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REFERENCES

1. S. Pissis, "Parallel Fourier transformations using shared memory nodes," MSc in High Performance Computing (Univ. Edinburgh, 2008).
2. A. V. Starovoytov, "On multidimensional analog of Cooley-Tukey algorithm," Bull. Siberian State Space Univ. Named after Academician M. F. Reshetnev, Issue 1 (27), 69–73 (2010).
3. V. S. Tutatchikov, O. I. Kiselev, and M. V. Noskov, "Calculating the n-dimensional fast Fourier transform," Pattern Recogn. Image Anal. **23** (3), 429–433 (2013).

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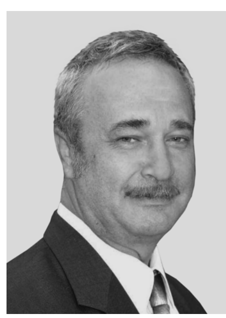

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